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The Physical Foundation of $F_N=kh^{3/2}$ for Conical/Pyramidal Indentation Loading Curves

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Summary: A physical deduction of the $F_N = kh^{3/2}$ relation (where F_N is normal force, k penetration resistance and h penetration depth) for conical/pyramidal indentation loading curves has been achieved on the basis of elementary mathematics. The indentation process couples the productions of volume and pressure to the displaced material that often partly plasticizes due to such pressure. As the pressure/plasticizing depends on the indenter volume, it follows that $F_N = F_p^{1/3} \cdot F_i^{2/3}$, where the index p stands for pressure/plasticizing and i for indentation. F_p does not contribute to the penetration, only F_i. The exponent 2/3 on F_i shows that experimentally F_N is applied but only $F_N^{2/3}$ is responsible for the penetration depth h. Thus, $F_N = kh^{3/2}$ is deduced and the physical reason is the loss of $F_N^{1/3}$ for the depth. Unfortunately, this has not been considered in teaching, textbooks, and the previous deduction of numerous common mechanical parameters, when the Love/Sneddon deductions of an exponent 2 on h were accepted and applied. The various unexpected experimental verifications and applications of the correct exponent 3/2 are mentioned and cited. Undue mechanical parameters require correction for safety reasons.

Key words: force-depth relation, nanoindentation loading curves, penetration resistance, physical deduction, pressure and plasticizing

Introduction

In 1939 Love and in 1965 Sneddon mathematically solved the "Boussinesq's problem" with different results, but both with the prediction of normal load (F_N) being proportional to depth square for conical and thus also pyramidal indentations. This has been widely accepted in publications and leading textbooks, and used for the deduction of various mechanical parameters that are still in use. Exponent 2 is also the result of numerous finite element simulations, when these use quadratic displacement elements (for example Wang et al. 2008; cf. Soare et al. 2005). Such simulations are often claimed to concur with published loading curves. However, more precise analysis reveals since 2004 that the experimental exponent is 3/2 instead. Simulated and experimental curves do not even correspond when published in the same paper. Only the analysis using the correct exponent can show, how to distinguish initial surface effects and phase changes under the load if these occur (Kaupp and Naimi-Jamal, 2004; Naimi-Jamal and Kaupp, 2005). The linear correlation coefficient for the slope k (F_N versus $h^{3/2}$) continues to always prove r >0.999 or for less noisy measurements r >0.9999 (Kaupp and Naimi-Jamal, 2010, and the cited more recent publications up to 2014). It was therefore possible to introduce the concept of penetration resistance for the safe comparison of materials' properties and compatibilities (Kaupp and Naimi-Jamal 2013), the energetic of indentations with the important finding that 80% of F_N is used for the indentation work and 20% for all the other force-induced energetic events (Kaupp 2013). Temperature dependent indentations even allow for the calculation of the activation energy of phase changes from nothing else than from indentation loading curves (Kaupp 2014). What's still missing was the physical reason for the experimentally verified successful exponent 3/2 on h, and this has been rightfully asked for by new-comers and experts in the field. Thus, the appreciation of the new exponent against textbooks (except Kaupp, 2006) requires the deduction of the exponent 3/2 on h. We report now on an unexpectedly short deduction of the physical reason that was not thought upon till now.

Experimental Background

The instrumental indentation experiment uses in most cases a diamond indenter that is continuously pressed with normal force (F_N) onto a level surface until the continuously recorded depth h is reached. By doing so, the volume V of the indenter is intruded and it shifts original material towards the bulk while producing pressure to it. Depending on the materials' properties such pressure p may persist (fully elastic) or it is partly released by some sort of plasticizing and migration with all of the known long-range effects. This scheme is principally equivalent with all of the different loading types normal to level surfaces and has been experimentally verified for all mechanisms of plasticizing (Kaupp and Naimi-Jamal 2013). Such retained pressure is, of course, used in unloading curves for the calculation of the elastic modulus, which does however not apply to the present topic. With this in mind we can start the deduction of the exponent 3/2.

Results and Discussion

The indentation couples two processes that must be differentiated, because the applied force must serve both of them. The production of volume is thus attributed to the fraction F_i^m for indentation, and the production of pressure + loss of pressure (by plasticizing via pressure) to the displaced material is attributed to the fraction F_p^n for pressure. As the multiplication of both factors must give the product F_N, these fractional forces must have exponents m and n <1, so that we obtain Equation (1).

$$F_N = F_i^{\,m} \cdot F_p^{\,n} \qquad (1)$$

For the determination of the exponents m and n we use the maximal pressure that could be reached at the depth h for absence of plasticizing. It is p + loss of p and we call it p_{max}. Equation (2) is evident, and the mathematical expression for a cone is V_{cone}.

$$p_{max} = K\,V; \quad V_{cone} = \pi\,(\tan\alpha)^2\,h^3/3 \qquad (2)$$

Equation (2) reveals that p_{max} and thus also F_p are proportional to h^3 of the cone. Formula (3) is thus obtained for cones and pyramids (with effective "effective cone angles" α.

$$p_{max} \propto h^3 \text{ and thus also } F_p \propto h^3 \qquad (3)$$

Formula (3) reveals the $F_P^{1/3}$ proportionality to the depth h, but $F_p^{1/3}$ does not contribute to the depth. Nevertheless, when n = 1/3, m must be 2/3 according to Equation (1), and this gives Equation (4).

$$F_N = F_i^{\,2/3} \cdot F_p^{\,1/3} \qquad (4)$$

The exponent 2/3 on F_i in Equation (4) reveals that instrumental indentation applies F_N**,** but only the fraction $F_N^{2/3}$ is responsible for the penetration and its depth h. This is expressed with Equation (5) that is thus physically deduced.

$$F_N = k\, h^{3/2} \tag{5}$$

The unavoidable pressure/plasticizing factor $F_p^{1/3}$ is lost for the depth. This is the physical reason for the exponent 3/2 on h instead of recently assumed 2 for cones and pyramids.

Conclusions

The physical deduction of the exponent 3/2 on h with elementary mathematics for indentation loading curves of cones and pyramids reveals a clear-cut physical reason. It will certainly strengthen the appreciation of exactly quantitative instrumental nano-, micro-, and macro-indentations. When required, the respective penetration resistance constant k ($N/m^{3/2}$) can be easily parameterized (see Equation 2). An example would be when a penetration resistance k shall be compared with different indenter geometries. But when the exponent on h of loading curves is used for hardness H, modulus E, or further parameter calculations, the correct exponent 3/2 should be used (but not 2 as for example at Oliver, 2001, and many others). Also the numerous recent plasticity parameters for biological materials in a tutorial of Oyen and Cook (2009) were deduced with the erroneous exponent 2 on h, and are thus subject to correcting re-deduction. Only the correct exponent allows for more advanced important applications that revealed unexpected materials' qualities. Some of these are named in the Introduction, others can be found in the cited papers of the author. Reliable mechanical qualities on the sound physical basis are most important for the proper adjustment of technical and medicinal composites and joints, for safety reasons. This is particularly important in the pressure range for phase changes, the onset of which can only be detected in the loading curves by analysis with the correct exponent 3/2 on h. It is hoped that all of that will now be better acknowledged in teaching and textbooks as well.

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I herewith confirm that I have no conflict of interest.

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